Solving Temporal Problems using SMT: Weak Controllability

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Temporal Problems with Uncertainty

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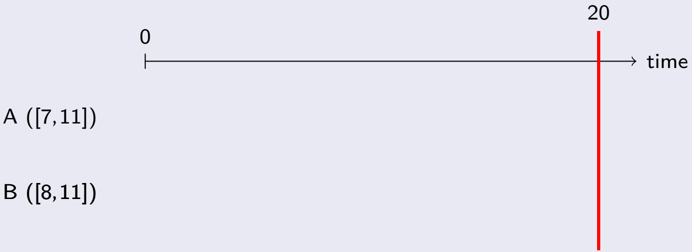
Activity View



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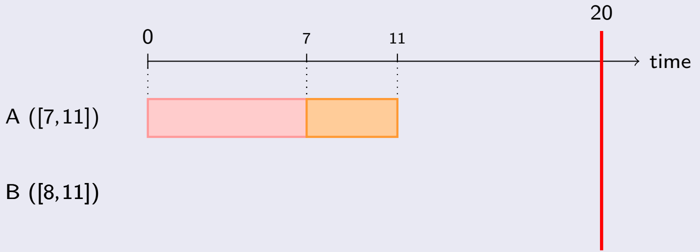

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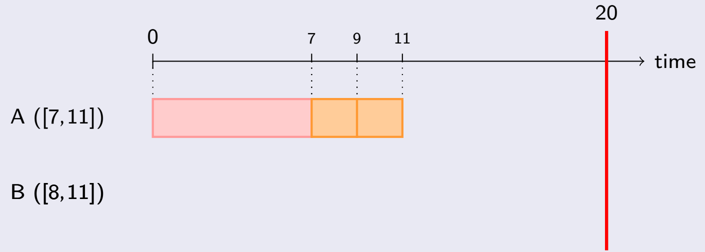

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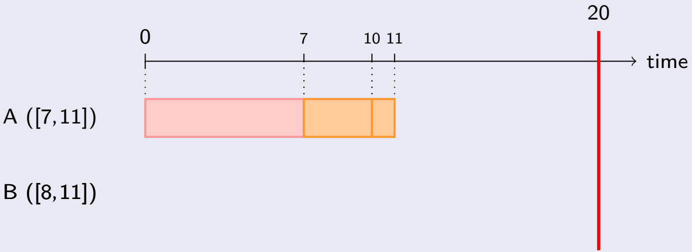

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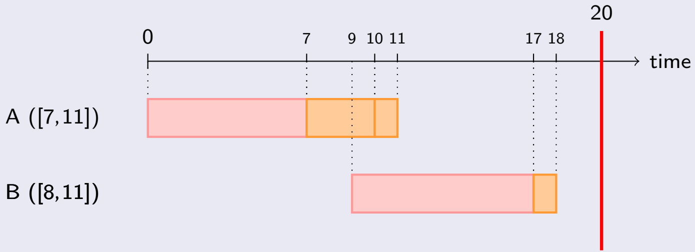

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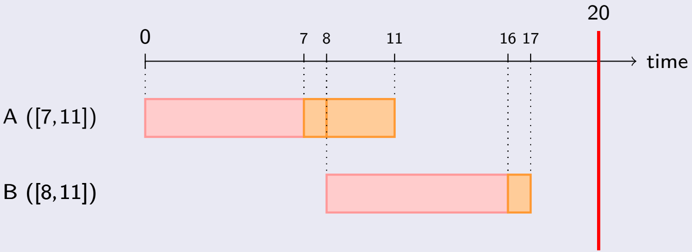

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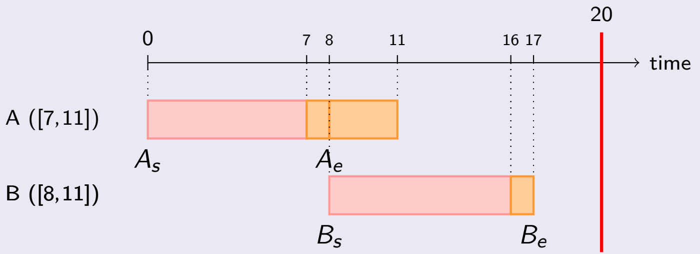

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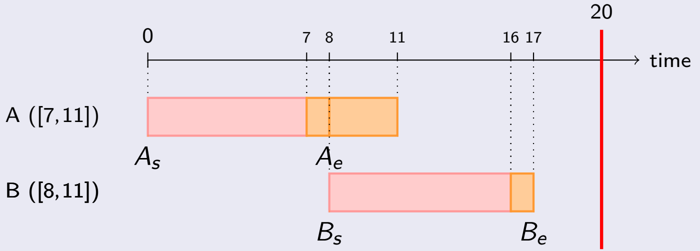
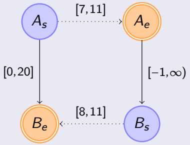


Temporal Problem View

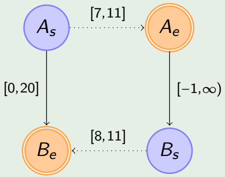


A_s, B_s are **Controllable Time Points**
A_e, B_e are **Uncontrollable Time Point**

⟶ represents **Free Constraints**
⋯→ represents **Contingent Constraints**

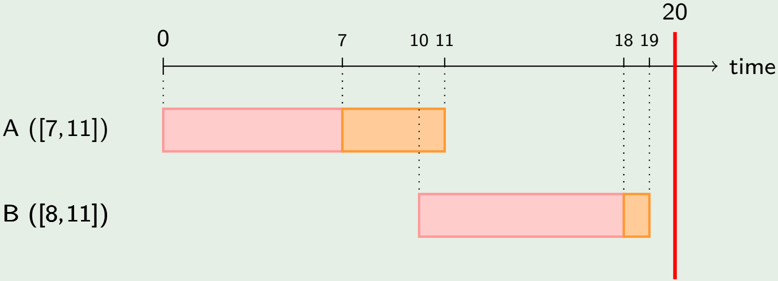
Schedules

Example



Fixed Schedule (Strong Controllability)

- $start(A)$ at 0
- $start(B)$ at 10



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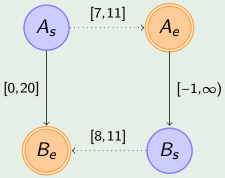
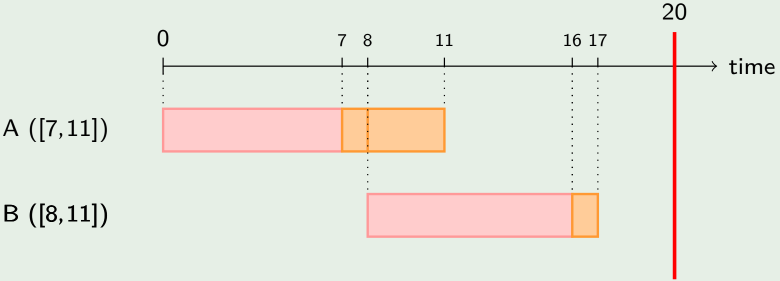


Dynamic Schedule (Dynamic Controllability)

- $start(A)$ at 0
- $start(B)$ at A_e



Schedules

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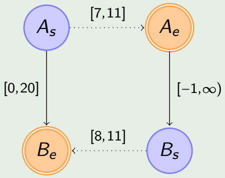
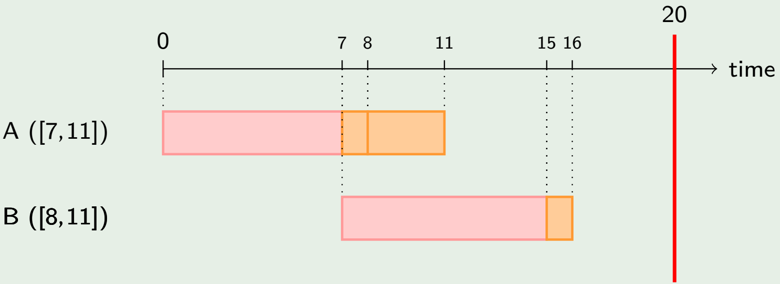


Clairvoyant Schedule (Weak Controllability)

- $start(A)$ at 0
- $start(B)$ at $A_e - 1$



Constraint Taxonomy

Notation

- X_c is the set of *controllable time points*
- X_u is the set of *uncontrollable time points*
- C_c is the set of *contingent constraints*
- C_f is the set of *free constraints*

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Type of constraint in C_f	**Problem class**
No disjunctions $(x_i - x_j) \in [l, u]$	STPU Simple Temporal Problem with Uncertainty
Interval disjunctions $(x_i - x_j) \in \bigcup_w [l_w, u_w]$	TCSPU Temporal Constraint Satisfaction Problem with Uncertainty
Arbitrary disjunctions $\bigvee_w ((x_{i_w} - x_{j_w}) \in [l_w, u_w])$	DTPU Disjunctive Temporal Problem with Uncertainty

In this paper: Weak Controllability

Definition

A Temporal Problem with Uncertainty is **Weakly Controllable** if and only if

$$\forall \vec{X}_u . \exists \vec{X}_c . (C_c(\vec{X}_c, \vec{X}_u) \rightarrow C_f(\vec{X}_c, \vec{X}_u))$$

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Strategies

A (clairvoyant) winning strategy is a function $f : \vec{X}_u \rightarrow \vec{X}_c$ such that

$$\forall \vec{X}_u . (C_c(f(\vec{X}_u), \vec{X}_u) \rightarrow C_f(f(\vec{X}_u), \vec{X}_u))$$

Contributions

1. **Decision procedure for Weak Controllability**
   - Formalization in Linear Real Arithmetic logic
   - Efficient encodings in Satisfiability Modulo Theory (*SMT*)
   - Not discussed in this talk

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   - Formalization in Linear Real Arithmetic logic
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2. **Strategy extraction algorithms for *STPU***
   - Algorithms for linear strategy extraction
   - Proof of non-linear strategy in general
   - Algorithms for piecewise linear strategy extraction

Outline

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Satisfiability Modulo Theory (*SMT*)

SMT is the problem of deciding satisfiability of a first-order Boolean combination of theory atoms in a given theory T.

A formula ϕ is satisfiable if there exists a model μ such that $\mu \models \phi$.

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$\exists y, z. (\forall x.(x > 0) \vee (y \geq x)) \wedge (z \geq y)$ is satisfiable in the theory of real arithmetic because

$$\mu = \{(y, 6), (z, 8)\}$$

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SMT procedures support various theories.

In this work:

- *LRA* (*Linear Real Arithmetic*)
- *QF_LRA* (*Quantifier-Free LRA*)

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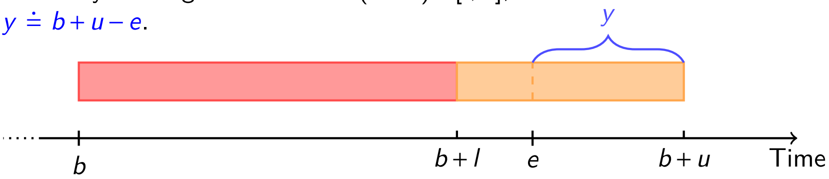
Weak Controllability definition immediately maps in *SMT* (*LRA*)

Uncontrollability Isolation

Let $e \in X_u$ and $b \in X_c$.
For every contingent constraint $(e - b) \in [l, u]$, we introduce an offset $y \doteq b + u - e$.



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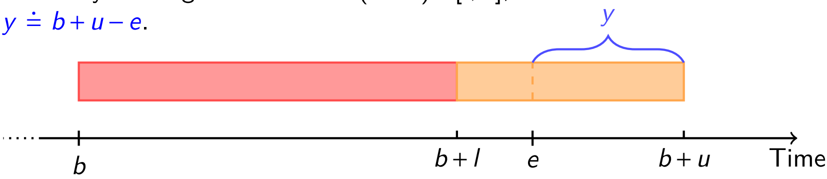

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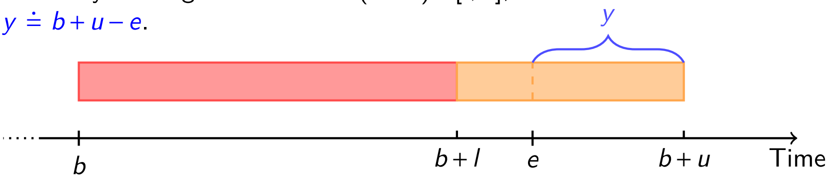

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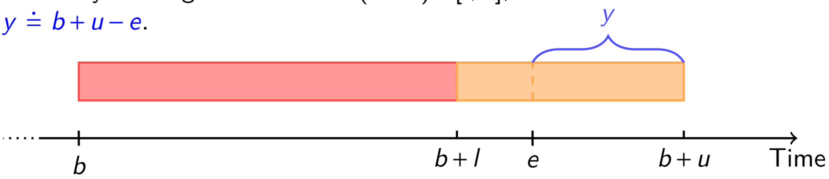

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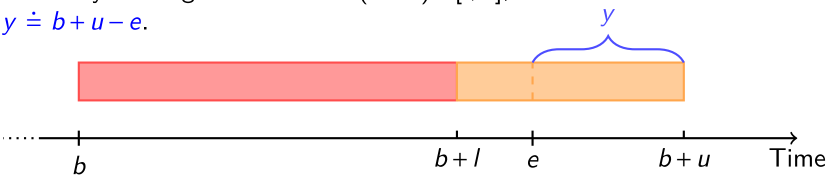

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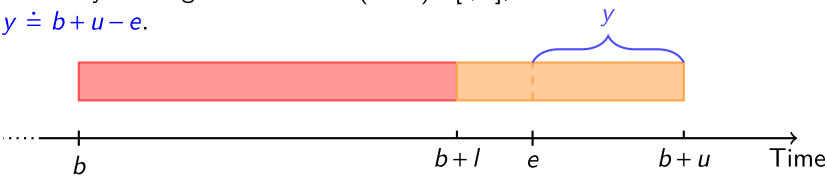

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- Weak Controllability becomes $\forall \vec{Y}_u . \exists \vec{X}_c . (\Gamma(\vec{Y}_u) \rightarrow \Psi(\vec{X}_c, \vec{Y}_u))$
- A winning strategy is now a function $f : \vec{Y}_u \rightarrow \vec{X}_c$

Strategies: Intuition

Given the solution space **P**, in the space of $\vec{X}_c$ and $\vec{Y}_u$, a strategy is a (possibly non-continuous) surface S, such that $P \cap S$ projected in the space of $\vec{Y}_u$ only, contains the polyhedron $\Gamma(\vec{Y}_u)$.

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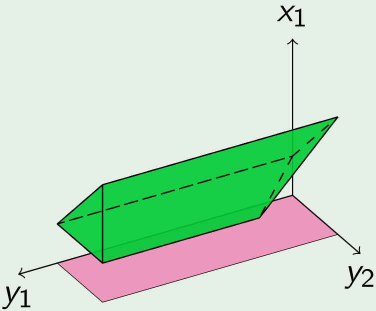

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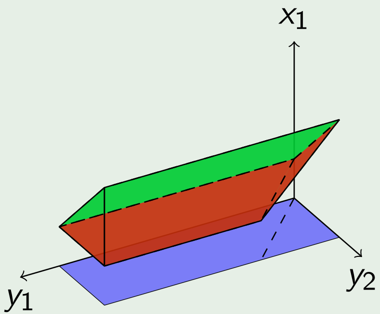

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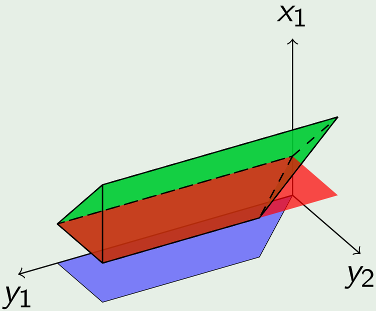

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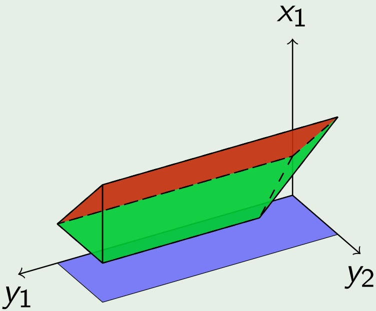

If S is a (hyper-)plane, the strategy is linear, i.e.

$$f(\vec{y}) \doteq A \cdot \vec{y} + \vec{b}$$

Outline

Linearity is not enough

Theorem

Not every weakly controllable STPU admits a linear strategy.

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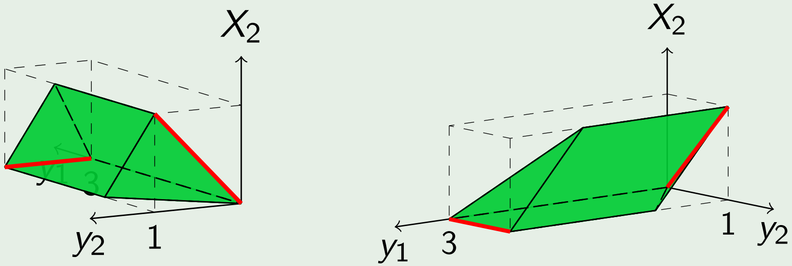
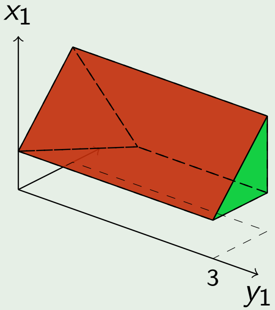


Encoding in *SMT* (*QF_LRA*)

(Another) Example



$$\Gamma(y_1, y_2) \doteq y_1 \geq 0 \land y_1 \geq 0 \land$$
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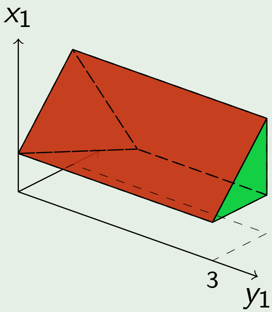

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$$f(\vec{y}) \doteq (a_1\ a_2) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + b$$

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$$\Gamma(y_1, y_2) \rightarrow \Psi(f(a_1, a_2, b), y_1, y_2)$$

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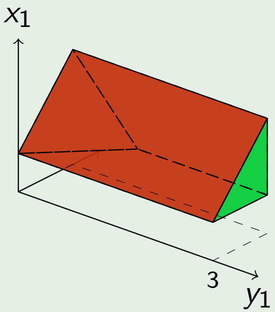

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$$Enc(a_1, a_2, c) \doteq$$

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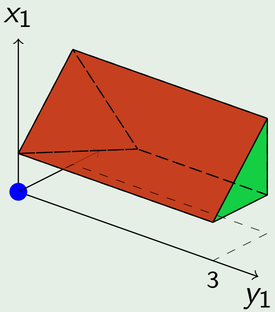

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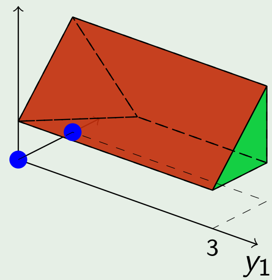

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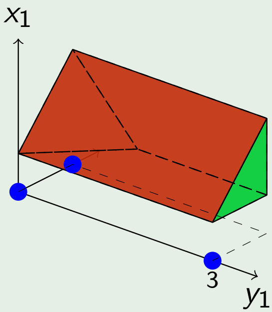

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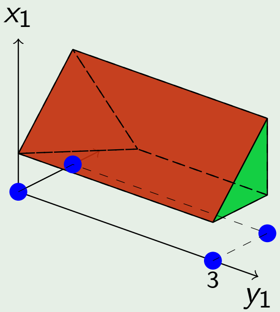

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Incremental Weakening

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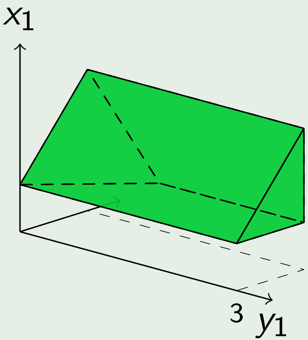

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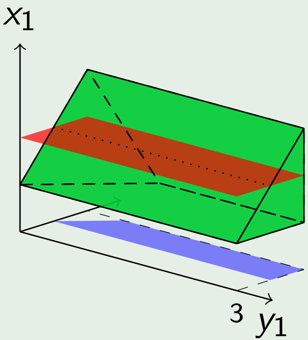
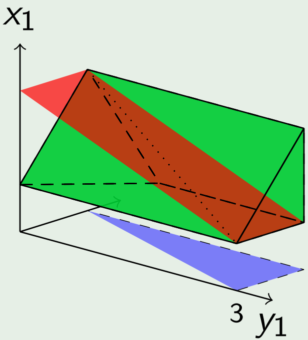


Observed uncontrollable offsets:

- $\emptyset$
- $\{y_1\}$
- $\{y_2\}$
- $\{y_1, y_2\}$

Incremental Weakening

Example



Observed uncontrollable offsets:

- $\emptyset$
- $\{y_1\}$
- $\{y_2\}$
- $\{y_1, y_2\}$

Incremental Weakening

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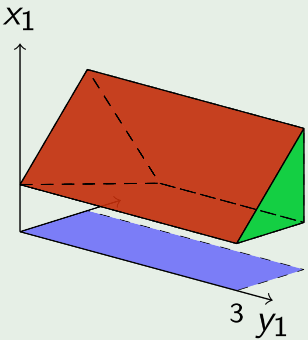

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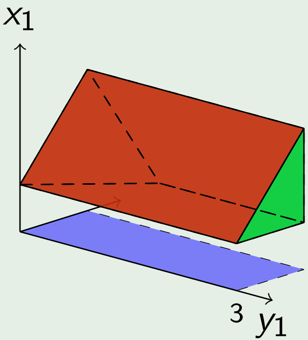

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Intuition

If we do not observe the *i*-th variable, the *i*-th column in the matrix is filled with 0.

Outline

Definitions

Piecewise linear strategies

f is a piecewise linear strategy if it has the form

$$\begin{aligned} f(\vec{y}) \doteq\ & \text{If } \phi_1(\vec{y}) \text{ then } A_1 \cdot \vec{y} + \vec{b}_1; \\ & \text{If } \phi_2(\vec{y}) \text{ then } A_2 \cdot \vec{y} + \vec{b}_2; \\ & \dots \\ & \text{If } \phi_n(\vec{y}) \text{ then } A_n \cdot \vec{y} + \vec{b}_n; \end{aligned}$$

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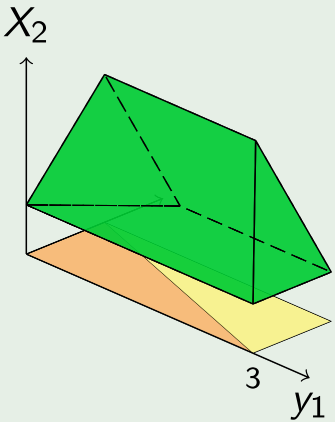
Simplexes

An n-simplex is an n-dimensional polytope which is the convex hull of its $n+1$ vertexes. E.g.

- 2-d $\rightarrow$ Triangle
- 3-d $\rightarrow$ Tetrahedron

Enumerating all the maximal simplexes

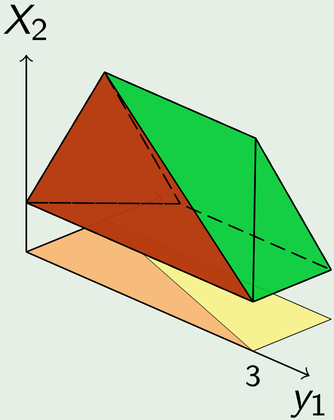
(Another) Example



Strategy:

Enumerating all the maximal simplexes

(Another) Example



Strategy:

$$\text{If } y_2 \leq -\frac{1}{3}y_1 + 1 \text{ then } S_1;$$

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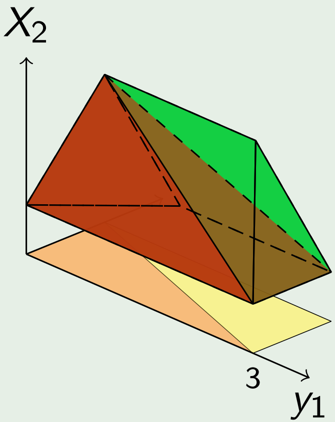

Strategy:

If $y_2 \leq -\frac{1}{3}y_1 + 1$ then S_1;

If $y_2 > -\frac{1}{3}y_1 + 1$ then S_2;

Lazy extraction

Idea

Pick a simplex $R(\vec{Y}_u)$ in $\Gamma(\vec{Y}_u)$, find a linear strategy $S(\vec{Y}_u)$ for $R(\vec{Y}_u)$, and remove the region where $S(\vec{Y}_u)$ is applicable from $\Gamma(\vec{Y}_u)$. Iterate until $\Gamma(\vec{Y}_u)$ is empty.

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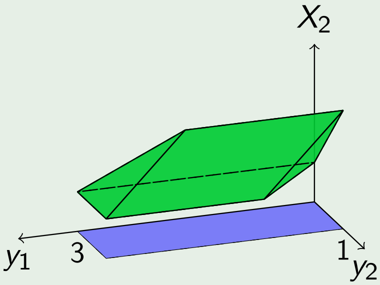

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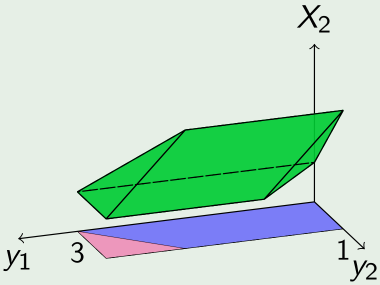

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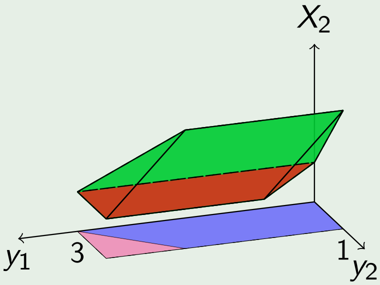

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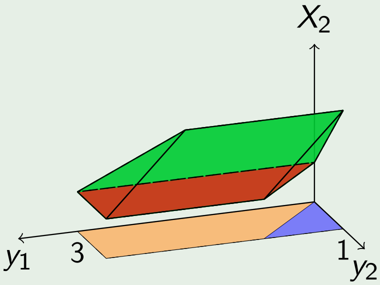

Strategy:

If $y_1 \geq y_2$ then S_1;

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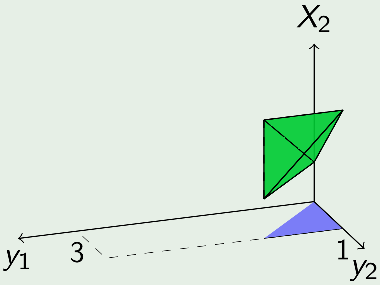

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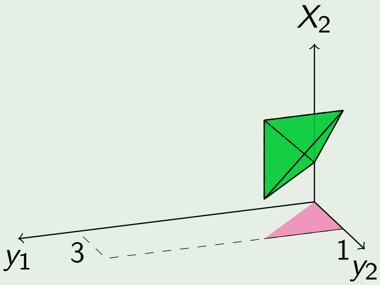

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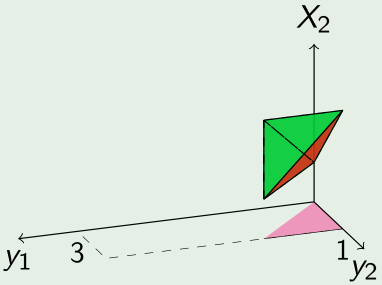


Strategy:

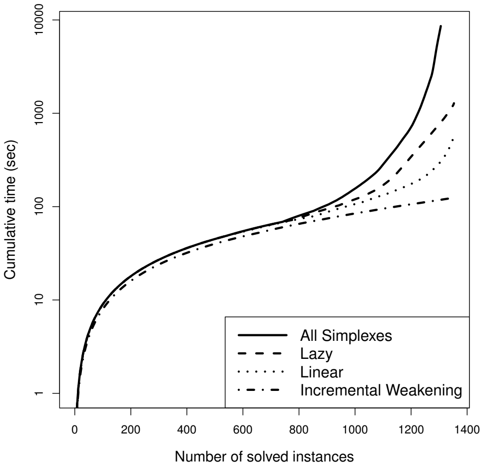
If $y_1 \geq y_2$ then S_1;

If $y_1 < y_2$ then S_2;

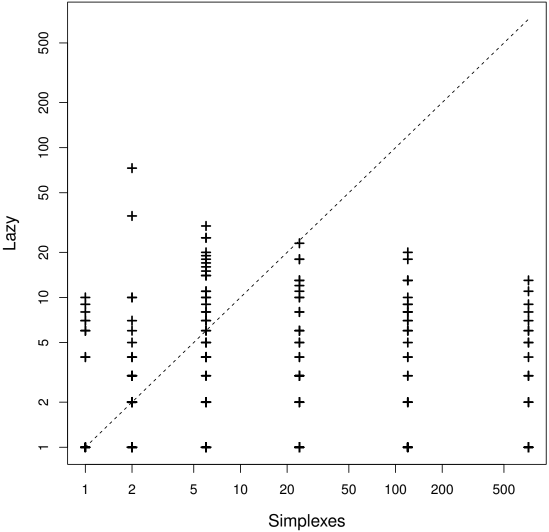
Outline

Scalability of strategy extraction algorithms

- Random instance generator derived from TSAT++ experiments
- Implementation
  - Python implementation
  - MathSAT5 API
- 4 algorithms
  - Linear
  - Incremental Weakening
  - All simplexes
  - Lazy
- 1354 weakly controllable instances admitting a linear strategy



Extracted strategy size



Outline

Conclusions

Contributions

- SMT-based approach for Weak Controllability in the general case
- Algorithms for *STPU* linear strategy extraction
- Proof of non-linear strategy existence
- Algorithms for *STPU* piecewise linear strategy extraction

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Future works

- Cost function optimization
  (Considering trade-off between linearity and optimality)
- Dynamic Controllability

Thanks

Thanks for your attention!